On the Fundamental Formulae of Dynamics.

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Formation of a new Indeterminate Formula of Motion by the Substitution of the Variations of the Components of Acceleration for the Variations of the Coordinates in the usual Formula.

The laws of motion are frequently expressed by an equation of the form

$$\Sigma\left[(X-m\ddot{x})\,\delta x+(Y-m\ddot{y})\,\delta y+(Z-m\ddot{z})\,\delta z\right]=0, \tag{1}$$

in which

m denotes the mass of a particle of the system considered,

$x,\ y,\ z$ its rectangular coordinates,

$\ddot{x},\ \ddot{y},\ \ddot{z}$ the second differential coefficients of the coordinates with respect to the time,

$X,\ Y,\ Z$ the components of the forces acting on the particle,

$\delta x,\ \delta y,\ \delta z$ any arbitrary variations of the coordinates which are simultaneously possible, and

Σ a summation with respect to all the particles of the system.

It is evident that we may substitute for δx, δy, δz any other expressions which are capable of the same and only of the same sets of simultaneous values.

Now if the nature of the system is such that certain functions A, B, etc. of the coordinates must be constant, or given functions of the time, we have

$$(2)\quad\left\{\begin{aligned}&\Sigma\left(\frac{dA}{dx}\,\delta x+\frac{dA}{dy}\,\delta y+\frac{dA}{dz}\,\delta z\right)=0,\\&\Sigma\left(\frac{dB}{dx}\,\delta x+\frac{dB}{dy}\,\delta y+\frac{dB}{dz}\,\delta z\right)=0,\\&\text{etc.}\end{aligned}\right.$$

These are the *equations of condition*, to which the variations in the general equation of motion (1) are subject. But if A is constant or a determined function of the time, the same must be true of $\dot{A}$ and $\ddot{A}$. Now

$$\dot{A}=\Sigma\left(\frac{dA}{dx}\,\dot{x}+\frac{dA}{dy}\,\dot{y}+\frac{dA}{dz}\,\dot{z}\right)$$

and

$$\ddot{A} = \Sigma \left(\frac{dA}{dx} \ddot{x} + \frac{dA}{dy} \ddot{y} + \frac{dA}{dz} \ddot{z} \right) + H,$$

where H represents terms containing only the second differential coefficients of A with respect to the coordinates, and the first differential coefficients of the coordinates with respect to the time. Therefore, if we conceive of a variation affecting the accelerations of the particles at the time considered, but not their positions or velocities, we have

$$(3) \left\{ \begin{array}{l} \delta \ddot{A} = \Sigma \left(\frac{dA}{dx} \delta \ddot{x} + \frac{dA}{dy} \delta \ddot{y} + \frac{dA}{dz} \delta \ddot{z} \right) = 0, \\ \text{and, in like manner,} \\ \delta \ddot{B} = \Sigma \left(\frac{dB}{dx} \delta \ddot{x} + \frac{dB}{dy} \delta \ddot{y} + \frac{dB}{dz} \delta \ddot{z} \right) = 0, \\ \text{etc.} \end{array} \right.$$

Comparing these equations with (2), we see that when the *accelerations* of the particles are regarded as subject to the variation denoted by δ, but not their positions or velocities, the possible values of $\delta \ddot{x}$, $\delta \ddot{y}$, $\delta \ddot{z}$ are subject to precisely the same restrictions as the values of δx, δy, δz, when the *positions* of the particles are regarded as variable. We may, therefore, write for the general equation of motion

$$(4) \qquad \Sigma \left[(X - m\ddot{x})\, \delta \ddot{x} + (Y - m\ddot{y})\, \delta \ddot{y} + (Z - m\ddot{z})\, \delta \ddot{z} \right] = 0,$$

regarding the positions and velocities of the particles as unaffected by the variation denoted by δ,—a condition which may be expressed by the equations

$$(5) \left\{ \begin{array}{lll} \delta x = 0, & \delta y = 0, & \delta z = 0, \\ \delta \dot{x} = 0, & \delta \dot{y} = 0, & \delta \dot{z} = 0. \end{array} \right.$$

We have so far supposed that the conditions which restrict the possible motions of the systems may be expressed by *equations* between the coordinates alone or the coordinates and the time. To extend the formula of motion to cases in which the conditions are expressed by the characters $\leqq$ or $\geqq$, we may write

$$(6) \qquad \Sigma \left[(X - m\ddot{x})\, \delta \ddot{x} + (Y - m\ddot{y})\, \delta \ddot{y} + (Z - m\ddot{z})\, \delta \ddot{z} \right] \leqq 0.$$

The conditions which determine the possible values of $\delta \ddot{x}$, $\delta \ddot{y}$, $\delta \ddot{z}$ will not, in such cases, be entirely similar to those which determine the possible values of δx, δy, δz, when the coordinates are regarded as variable. Nevertheless,

the laws of motion are correctly expressed by the formula (6), while the formula

$$\Sigma \left[(X - m\ddot{x})\, \delta x + (Y - m\ddot{y})\, \delta y + (Z - m\ddot{z})\, \delta z \right] \leqq 0, \tag{7}$$

does not, as naturally interpreted, give so complete and accurate an expression of the laws of motion.

This may be illustrated by a simple example.

Let it be required to find the acceleration of a material point, which, at a given instant, is moving with given velocity on the frictionless surface of a body (which it cannot penetrate, but which it may leave), and is acted on by given forces. For simplicity, we may suppose that the normal to the surface, drawn outward from the moving point at the moment considered, is parallel to the axis of X and in the positive direction. The only restriction on the values of δx, δy, δz is that

$$\delta x \geqq 0.$$

Formula (7) will therefore give

$$\ddot{x} \geqq \frac{X}{m}, \qquad \ddot{y} = \frac{Y}{m}, \qquad \ddot{z} = \frac{Z}{m}.$$

The condition that the point shall not penetrate the body gives another condition for the value of $\ddot{x}$. If the point remains upon the surface, $\ddot{x}$ must have a certain value N, determined by the form of the surface and the velocity of the point. If the value of $\ddot{x}$ is less that this, the point must penetrate the body. Therefore,

$$\ddot{x} \geqq N.$$

But this does not suffice to determine the acceleration of the point.

Let us now apply formula (6) to the same problem. Since $\ddot{x}$ cannot be less than N,

$$\text{if } \ddot{x} = N, \quad \delta\ddot{x} \geqq 0.$$

This is the only restriction on the value of $\delta\ddot{x}$, for if $\ddot{x} > N$, the value of $\delta\ddot{x}$ is entirely arbitrary. Formula (6), therefore, requires that

$$\text{if } \ddot{x} = N, \quad \ddot{x} \geqq \frac{X}{m};$$

$$\text{but if } \ddot{x} > N, \quad \ddot{x} = \frac{X}{m}:$$

—that is, (since $\ddot{x}$ cannot be less than N), that $\ddot{x}$ shall be equal to the greater of the quantities N and $\frac{X}{m}$, or to both, if they are equal,—and that

$$\ddot{y} = \frac{Y}{m}, \quad \ddot{z} = \frac{Z}{m}.$$

The values of $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ are therefore entirely determined by this formula in connection with the conditions afforded by the constraints of the system.*

The following considerations will show that what is true in this case is also true in general, when the conditions to which the system is subject are such that certain functions of the coordinates cannot exceed certain limits, either constant or variable with the time. If certain values of $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ (with unvaried values of x, y, z, and $\dot{x}$, $\dot{y}$, $\dot{z}$) are simultaneously possible at a given instant, equal or proportional values with the same signs, must be possible for δx, δy, δz immediately after the instant considered, and must satisfy formula (1), and therefore (6), in connection with the values of $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, X, Y, Z immediately after that instant. The values of $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, thus determined, are of course the very quantities which we wish to obtain, since the acceleration of a point at a given instant does not denote anything different from its acceleration immediately after that instant.

For an example of a somewhat different class of cases, we may suppose that in a system, otherwise free, x cannot have a negative value. Such a condition does not seem to affect the possible values of δx, as naturally interpreted in a dynamical problem. Yet, if we should regard the value of δx in (7) as arbitrary, we should obtain

$$\ddot{x} = \frac{X}{m},$$

which might be erroneous. But if we regard δx as expressing a velocity of which the system, if at rest, would be capable, (which is not a natural signification of the expression,) we should have $\delta x \geqq 0$, which, with (7), gives

$$\ddot{x} \geqq \frac{X}{m}.$$

This is not incorrect, but it leaves the acceleration undetermined. If we should regard δx as denoting such a variation of the velocity as is possible for the system when it has its given velocity (this also is not a natural

* The failure of the formula (7) in this case is rather apparent than real; for, although the formula apparently allows to $\ddot{x}$, at the instant considered, a value exceeding both N and $\frac{X}{m}$, it does not allow this for any interval, however short. For if $\ddot{x} < N$, the point will immediately leave the surface, and then the formula requires that $\ddot{x} = \frac{X}{m}$.

signification of the expression), formula (7) would give the correct value of $\ddot{x}$ except when $\dot{x} = 0$. In this case (which cannot be regarded as exceptional in a problem of this kind), we should have $\delta x \geqq 0$, which will leave $\ddot{x}$ undetermined, as before.

The application of formula (6), in problems of this kind, presents no difficulty. From the condition

$$\dot{x} \geqq 0,$$

we obtain, first, $\qquad$ if $\dot{x} = 0, \quad \ddot{x} \geqq 0,$

then, $\qquad$ if $\dot{x} = 0 \quad$ and $\quad \ddot{x} = 0, \quad \delta\ddot{x} \geqq 0,$

which is the only limitation on the value of $\delta\ddot{x}$. With this condition, we deduce from (6) that either

$$\dot{x} = 0, \quad \ddot{x} = 0, \quad \text{and} \quad \ddot{x} \geqq \frac{X}{m};$$

or

$$\ddot{x} = \frac{X}{m}.$$

That is, if $\dot{x} = 0$, $\ddot{x}$ has the greater of the values $\frac{X}{m}$ and 0; otherwise, $\ddot{x} = \frac{X}{m}$.

In cases of this kind also, in which the function which cannot exceed a certain value involves the velocities (with or without the coordinates), one may easily convince himself that formula (6) is always valid, and always sufficient to determine the accelerations with the aid of the conditions afforded by the constraints of the system.

But instead of examining such cases in detail, we shall proceed to consider the subject from a more general point of view.

Comparison of the New Formula with the Statical Principle of Virtual Velocities.—Case of Discontinuous Changes of Velocity.

Formula (1) has so far served as a point of departure. The general validity of this, the received form of the indeterminate equation of motion, being assumed, it has been shown that formula (6) will be valid and sufficient, even in cases in which both (1) and (7) fail. We now proceed to show that the statical principle of *virtual velocities*, when its real signification is carefully considered, leads directly to formula (6), or to an analogous formula for the determination of the discontinuous changes of velocity, when such

occur. This will be the case even if we start with the usual analytical expression of the principle

(8) $$\Sigma\,(X\delta x + Y\delta y + Z\delta z) \leqq 0,$$

to which, at first sight, formula (6) appears less closely related than (7). For the variations of the coordinates in this formula must be regarded as relating to differences between the configuration which the system has at a certain time, and which it will continue to have in case of equilibrium, and some other configuration which the system might be supposed to have at some subsequent time. These temporal relations are not indicated explicitly in the notation, and should not be, since the statical problem does not involve the time in any quantitative manner. But in a dynamical problem, in which we take account of the time, it is hardly natural to use δx, δy, δz in the same sense. In any problem in which x, y, z are regarded as functions of the time, δx, δy, δz are naturally understood to relate to differences between the configuration which the system has at a certain time, and some other configuration which it might (conceivably) have had at that time *instead of* that which it actually had.

Now when we suppose a point to have a certain position, specified by x, y, z, at a certain time, its position at that time is no longer a subject of hypothesis or of question. It is its future positions which form the subject of inquiry. Its position in the immediate future is naturally specified by

$$x + \dot{x}dt + \frac{1}{2}\,\ddot{x}dt^2 + \text{etc.}, \quad y + \dot{y}dt + \frac{1}{2}\,\ddot{y}dt^2 + \text{etc.}, \quad z + \dot{z}dt + \frac{1}{2}\,\ddot{z}dt^2 + \text{etc.},$$

and we may regard the variations of these expressions as corresponding to the δx, δy, δz of the statical problem. It is evidently sufficient to take account of the first term of these expressions of which the variation is not zero. Now, x, y, z, as has already been said, are to be regarded as constant. With respect to the terms containing $\dot{x}$, $\dot{y}$, $\dot{z}$, two cases are to be distinguished, according as there is, or is not, a finite change of velocity at the instant considered.

Let us first consider the most important case, in which there is no discontinuous change of velocity. In this case, $\dot{x}$, $\dot{y}$, $\dot{z}$ are not to be regarded as variable (by δ), and the variations of the above expressions are represented by

$$\frac{1}{2}\,\delta\ddot{x}\,dt^2, \quad \frac{1}{2}\,\delta\ddot{y}\,dt^2, \quad \frac{1}{2}\,\delta\ddot{z}\,dt^2,$$

which are, therefore, to be substituted for δx, δy, δz in the general formula of equilibrium (8) to adapt it to the conditions of a dynamical problem. By this substitution (in which the common factor $\frac{1}{2}\,dt^2$ may of course be omitted), and the addition of the terms expressing the reaction against acceleration, we obtain formula (6).

But if the circumstances are such that there is (or may be) a discontinuity in the values of $\dot{x}$, $\dot{y}$, $\dot{z}$ at the instant considered, it is necessary to distinguish the values of these expressions before and after the abrupt change. For this purpose, we may apply $\dot{x}$, $\dot{y}$, $\dot{z}$ to the original values, and denote the changed values by $\dot{x} + \Delta\dot{x}$, $\dot{y} + \Delta\dot{y}$, $\dot{z} + \Delta\dot{z}$. The value of x at a time very shortly subsequent to the instant considered, will be expressed by $x + (\dot{x} + \Delta\dot{x})\,dt +$ etc., in which we may regard $\Delta\dot{x}$ as subject to the variation denoted by δ. The variation of the expression is therefore $\delta\Delta\dot{x}\,dt$. Instead of $-m\ddot{x}$, which expresses the reaction against acceleration, we need in the present case $-\Delta\dot{x}$ to express the reaction against the abrupt change of velocity. A reaction against such a change of velocity is, of course, to be regarded as infinite in intensity in comparison with reactions due to acceleration, and ordinary forces (such as cause acceleration) may be neglected in comparison. If, however, we conceive of the system as acted on by impulsive forces, (*i. e.* such as have no finite duration, but are capable of producing finite changes of velocity, and are measured numerically by the discontinuities of velocity which they produce in the unit of mass,) these forces should be combined with the reactions due to the discontinuities of velocity in the general formula which determines these discontinuities. If the impulsive forces are specified by X, Y, Z, the formula will be

$$(9) \qquad \Big[(\mathrm{X} - m\Delta\dot{x})\,\delta\Delta\dot{x} + (\mathrm{Y} - m\Delta\dot{y})\,\delta\Delta\dot{y} + (\mathrm{Z} - m\Delta\dot{z})\,\delta\Delta\dot{z}\Big] \leqq 0.$$

The reader will remark the strict analogy between this formula and (6), which would perhaps be more clearly exhibited if we should write $\frac{d\dot{x}}{dt}$, $\frac{d\dot{y}}{dt}$, $\frac{d\dot{z}}{dt}$ for $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ in that formula.

But these formulae may be established in a much more direct manner. For the formula (8), although for many purposes the most convenient expression of the principle of virtual velocities, is by no means the most convenient for our present purpose. As the usual name of the principle implies, it holds

true of velocities as well as of displacements, and is perhaps more simple and more evident when thus applied.*

If we wish to apply the principle, thus understood, to a moving system so as to determine whether certain changes of velocity specified by $\Delta\dot{x}$, $\Delta\dot{y}$, $\Delta\dot{z}$ are those which the system will really receive at a given instant, the velocities to be multiplied into the forces and reactions in the most simple application of the principle are manifestly such as may be imagined to be compounded with the assumed velocities, and are therefore properly specified by $\delta\Delta\dot{x}$, $\delta\Delta\dot{y}$, $\delta\Delta\dot{z}$. The formula (9) may therefore be regarded as the most direct application of the principle of virtual velocities to discontinuous changes of velocity in a moving system.

In the case of a system in which there are no discontinuous changes of velocity, but which is subject to forces tending to produce accelerations, when we wish to determine whether certain accelerations, specified by $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, are such as the system will really receive, it is evidently necessary to consider whether any possible variation of these accelerations is favored more than it is opposed by the forces and reactions of the system. The formula (7) expresses a criterion of this kind in the most simple and direct manner. If we regard a force as a tendency to increase a quantity expressed by $\ddot{x}$, the product of the force by $\delta\ddot{x}$ is the natural measure of the extent to which this tendency is satisfied by an arbitrary variation of the accelerations. The principle expressed by the formula may not be very accurately designated by the words *virtual velocities*, but it certainly does not differ from the principle of virtual velocities (in the stricter sense of the term), more than this differs from that of virtual displacements,—a difference so slight that the distinction of the names is rarely insisted upon, and that it is often very difficult to tell which

*Even in Statics, the principle of virtual *velocities*, as distinguished from that of virtual *displacements*, has a certain advantage in respect of its evidence. The demonstration of the principle in the first section of the *Mécanique Analytique*, if velocities had been considered instead of displacements, would not have been exposed to an objection, which has been expressed by M. Bertrand in the following words: "On a objecté, avec raison, à cette assertion de Lagrange l'exemple d'un point pesant en équilibre au sommet le plus élevé d'une courbe; il est évident qu'un déplacement infiniment petit le ferait descendre, et, pourtant, ce déplacement ne se produit pas." (*Mécanique Analytique*, troisième édition, tome 1, page 22, note de M. Bertrand.) The value of z (the height of the point above a horizontal plane) can certainly be diminished by a displacement of the point, but value of $\dot{z}$ is not affected by any velocity given to the point.

The real difficulty in the consideration of displacements is that they are only possible at a time subsequent to that in which the system has the configuration to which the question of equilibrium relates. We may make the interval of time infinitely short, but it will always be difficult, in the establishing of fundamental principles, to treat a conception of this kind (relating to what is possible after an infinitesimal interval of time) with the same rigor as the idea of velocities or accelerations, which, in the cases to which (9) and (6) respectively relate, we may regard as communicated immediately to the system.

form of the principle is especially intended, even when the principle is enunciated or discussed somewhat at length.

But, although the formulae (7) and (9) differ so little from the ordinary formulae, they not only have a marked advantage in respect of precision and accuracy, but also may be more satisfactory to the mind, in that the changes considered (to which δ relates), are not so violently opposed to all the possibilities of the case as are those which are represented by the variations of the coordinates.* Moreover, as we shall see, they naturally lead to various important laws of motion.

Transformation of the New Formula.

Let us now consider some of the transformations of which our general formula (7) is capable. If we separate the terms containing the masses of the particles from those which contain the forces, we have

$$(10)\qquad \Sigma\,(X\delta\ddot{x} + Y\delta\ddot{y} + Z\delta\ddot{z}) - \Sigma\left[\frac{1}{2}\,m\,\delta\,(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2)\right] \leqq 0\,,$$

or, if we write u for the acceleration of a particle,

$$(11)\qquad \Sigma\,(X\delta\ddot{x} + Y\delta\ddot{y} + Z\delta\ddot{z}) - \delta\,\Sigma\left(\frac{1}{2}\,mu^2\right) \leqq 0\,.$$

If, instead of terms of the form $X\delta\dot{x}$, or in addition to such terms, equation (1) had contained terms of the form $P\delta p$, in which p denotes any quantity determined by the configuration of the system, it is evident that these would give terms of the form $P\delta\ddot{p}$ in (7), (10) and (11). For the considerations which justified the substitution of $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ for δx, δy, δz in the usual formula

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* It may have seemed to some readers of the *Mécanique Analytique*—a work of which the unity of method is one of the most striking characteristics, and that to which its universally recognized artistic merit is in great measure due—that the treatment of dynamical problems in that work is not entirely analogous to the treatment of statical problems. The statical question, whether a system will remain in equilibrium in a given configuration, is determined by Lagrange by considering all possible motions of the system and inquiring whether there is any reason why the system should take any one of them. A similar method in dynamics would be based upon a comparison of a proposed motion with all other motions of which the system is capable without violating its kinematical conditions. Instead of this, Lagrange virtually reduces the dynamical problem to a statical one, and considers, not the possible variations of the proposed motion, but the motions which would be possible if the system were at rest. This reduction of a given problem to a simpler one, which has already been solved, is a method which has its advantages, but it is not the characteristic method of the *Mécanique Analytique*. That which most distinguishes the plan of this treatise from the usual type is the direct application of the general principle to each particular case.

The point is perhaps of small moment, and may be differently regarded by others, but it is mentioned here because it was a feeling of this kind (whether justified or not) and the desire to express the formula of motion by means of a maximum or minimum condition, in which the conditions under which the maximum or minimum subsists should be such as the problem naturally affords, (Gauss's principle of *least constraint* being at the time unknown to the present writer, and the conditions under which the minimum subsists in the principle of *least action* being such that that is hardly satisfactory as a fundamental principle,) which led to the formulae proposed in this paper.

were in no respect dependent upon the fact that x, y, z denote rectangular coordinates, but would apply equally to any other quantities which are determined by the configuration of the system.

Hence, if the moments of all the forces of the system are represented by the sum
$$\mathfrak{S}\,(Pdp)\,,$$
the general formula of motion may be written

$$\mathfrak{S}\,(P\delta\ddot{p}) - \delta\,\Sigma\left(\frac{1}{2}\,mu^2\right) \overline{\overline{<}}\; 0\,. \tag{12}$$

If the forces admit of a force-function V, we have

$$\delta\ddot{V} - \delta\,\Sigma\left(\frac{1}{2}\,mu^2\right) \overline{\overline{<}}\; 0\,,$$

or

$$\delta\left[\ddot{V} - \Sigma\left(\frac{1}{2}\,mu^2\right)\right] \overline{\overline{<}}\; 0\,. \tag{13}$$

But if the forces are determined in any way whatever by the configuration and velocities of the system, with or without the time, X, Y, Z and P will be unaffected by the variation denoted by δ, and we may write the formula of motion in the form

$$\delta\Sigma\left(X\ddot{x} + Y\ddot{y} + Z\ddot{z} - \frac{1}{2}\,mu^2\right) \overline{\overline{<}}\; 0\,, \tag{14}$$

or

$$\delta\left[\mathfrak{S}\,(P\ddot{p}) - \Sigma\left(\frac{1}{2}\,mu^2\right)\right] \overline{\overline{<}}\; 0\,. \tag{15}$$

If the forces are determined by the configuration alone, or the configuration and the time, $\delta X = 0$, $\delta Y = 0$, $\delta Z = 0$, $\delta P = 0$, and the general formula may be written

$$\delta\left[\frac{d}{dt}\,\Sigma\,(X\dot{x} + Y\dot{y} + Z\dot{z}) - \Sigma\left(\frac{1}{2}\,mu^2\right)\right] \overline{\overline{<}}\; 0\,, \tag{16}$$

or

$$\delta\left[\frac{d}{dt}\,\mathfrak{S}\,(P\dot{p}) - \Sigma\left(\frac{1}{2}\,mu^2\right)\right] \overline{\overline{<}}\; 0 \tag{17}$$

The quantity affected by δ in any one of the last five formulae has not only a maximum value, but absolutely the greatest value consistent with the constraints of the system. This may be shown in reference to (15) by giving to $\ddot{p}$, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, contained explicitly or implicitly in the expression affected by δ, any possible finite increments $\ddot{p}'$, $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$, and subtracting the original value of the expression from the value thus modified. Now,

$$\mathfrak{S}[P(\ddot{p}+\ddot{p}')] - \Sigma\left[\frac{1}{2}m\{(\ddot{x}+\ddot{x}')^2+(\ddot{y}+\ddot{y}')^2+(\ddot{z}+\ddot{z}')^2\}\right] - \mathfrak{S}(P\ddot{p}) + \Sigma\left[\frac{1}{2}m(\ddot{x}^2+\ddot{y}^2+\ddot{z}^2)\right]$$
$$= \mathfrak{S}(P\ddot{p}') - \Sigma\,[m\,(\ddot{x}\ddot{x}'+\ddot{y}\ddot{y}'+\ddot{z}\ddot{z}')] - \Sigma\left[\frac{1}{2}\,m\,(\ddot{x}'^2+\ddot{y}'^2+\ddot{z}'^2)\right]$$

But since $\ddot{p}'$, $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$ are proportional to and of the same sign with possible values of $\delta\ddot{p}$, $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$, we have, by the general formula of motion,

$$\mathfrak{S}\,(P\ddot{p}') - \Sigma\left[m\,(xx' + yy' + zz')\right] \leqq 0\,.$$

The second member of the preceding equation is therefore negative. The first member is therefore negative, which proves the proposition with respect to (15). The demonstration is precisely the same with respect to (13) and (14), which may be regarded as particular cases of (15).

To show the same with regard to (16) and (17), we have only to observe that the quantities affected by δ in these formulae differ from those affected by the same symbol in (14) and (15) only by the terms

$$\Sigma\,(\dot{X}x + \dot{Y}y + \dot{Z}z) \quad \text{and} \quad \mathfrak{S}\,(\dot{P}p)\,,$$

which will not be affected by any change in the accelerations of the system.

When the forces are determined by the configuration (with or without the time), the principle may be enunciated as follows: The accelerations in the system are always such that the acceleration of the rate of work done by the forces diminished by one-half the sum of the products of the masses of the particles by the squares of their accelerations has the greatest possible value.

The formula (17), although in appearance less simple than (15), not only is more easily enunciated in words, but has the advantage that the quantity $\frac{d}{dt}\,\mathfrak{S}\,(P\dot{p})$ is entirely determined by the system with its forces and motions, which is not the case with $\mathfrak{S}\,(P\ddot{p})$. The value of the latter expression depends upon the manner in which we choose to represent the forces. For example, if a material point is revolving in a circle under the influence of a central force, we may write either $X\ddot{x} + Y\ddot{y} + Z\ddot{z}$ or $R\ddot{r}$ for $P\ddot{p}$, R and r denoting respectively the force and radius vector. Now $X\ddot{x} + Y\ddot{y} + Z\ddot{z}$ is manifestly unequal to $R\ddot{r}$. But $X\dot{x} + Y\dot{y} + Z\dot{z}$ is equal to $R\dot{r}$, and $\frac{d}{dt}\,(X\dot{x} + Y\dot{y} + Z\dot{z})$ is equal to $\frac{d}{dt}\,(R\dot{r})$.

It may not be without interest to see what shape our general formulae will take in one of the most important cases of forces dependent upon the velocities. If a body which can be treated as a point is moving in a medium which presents a resistance expressed by any function of the velocity, the terms due to that resistance in the general formula of motion may be expressed in the form

$$\delta\left[\phi\,(v)\,\frac{\dot{x}}{v}\,\ddot{x} + \phi\,(v)\,\frac{\dot{y}}{v}\,\ddot{y} + \phi\,(v)\,\frac{\dot{z}}{v}\,\ddot{z}\right],$$

where v denotes the velocity and $\phi\ (v)$ the resistance. But

$$\frac{\dot{x}\ddot{x}}{v} + \frac{\dot{y}\ddot{y}}{v} + \frac{\dot{z}\ddot{z}}{v} = \frac{dv}{dt} = \dot{v}\,.$$

The terms due to the resistance reduce, therefore, to

$$\delta\,[\phi\,(v)\,\dot{v}]\,,$$

or,

$$\delta\,\frac{d}{dt}\,f\,(v)\,,$$

where f denotes the primitive of the function denoted by ϕ.

Discontinuous Changes of Velocity.—Formula (9), which relates to discontinuous changes of velocity, is capable of similar transformations. If we set

$$w^2 = \Delta\dot{x}^2 + \Delta\dot{y}^2 + \Delta\dot{z}^2,$$

the formula reduces to

$$\text{(18)} \qquad \delta\,\Sigma\,\left(\mathrm{X}\Delta\dot{x} + \mathrm{Y}\Delta\dot{y} + \mathrm{Z}\Delta\dot{z} - \frac{1}{2}\,mw^2\right) \leqq 0\,,$$

where X, Y, Z are to be regarded as constant. If $\mathfrak{S}\ (\mathrm{P}dp)$ represents the sum of the moments of the impulsive forces, and we regard P as constant, we have

$$\text{(19)} \qquad \delta\left[\mathfrak{S}\ (\mathrm{P}\Delta\dot{p}) - \Sigma\left(\frac{1}{2}\,mw^2\right)\right] \leqq 0\,.$$

The expressions affected by δ in these formulae have a greater value than they would receive from any other changes of velocity consistent with the constraints of the system.

Deduction of other Properties of Motion.

The principles which have been established furnish a convenient point of departure for the demonstration of various properties of motion relating to *maxima* and *minima*. We may obtain several such properties by considering how the accelerations of a system, at a given instant, will be modified by changes of the forces or of the constraints to which the system is subject. Let us suppose that the forces X, Y, Z of a system receive the increments X', Y', Z', in consequence of which, and of certain additional constraints, which do not produce any discontinuity in the velocities, the components of acceleration $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ receive the increments $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$. The expression

$$\text{(20)} \quad \Sigma\left[(X+X')(\ddot{x}+\ddot{x}')(Y+Y')(\ddot{y}+\ddot{y}') + (Z+Z')(\ddot{z}+\ddot{z}')\right.$$
$$\left. - \frac{1}{2}\,m\left\{(\ddot{x}+\ddot{x}')^2 + (\ddot{y}+\ddot{y}')^2 + (\ddot{z}+\ddot{z}')^2\right\}\right]$$

will be the greatest possible for any values of $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$ consistent with the constraints. But this expression may be divided into three parts,

(21) $$\Sigma\left[(X+X')\,\ddot{x}+(Y+Y')\,\ddot{y}+(Z+Z')\,\ddot{z}-\frac{1}{2}\,m\,(\ddot{x}^2+\ddot{y}^2+\ddot{z}^2)\right],$$

(22) $$\Sigma\,[X\ddot{x}'+Y\ddot{y}'+Z\ddot{z}'-m\,(\ddot{x}\ddot{x}'+\ddot{y}\ddot{y}'+\ddot{z}\ddot{z}')],$$

and

(23) $$\Sigma\left[X'\ddot{x}'+Y'\ddot{y}'+Z'\ddot{z}'-\frac{1}{2}\,m\,(\ddot{x}'^2+\ddot{y}'^2+\ddot{z}'^2)\right].$$

The first part is evidently constant with reference to variations of $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$, and may, therefore, be neglected. With respect to the second part, we observe that by the general formula of motion we have

$$\Sigma\,[X\delta\ddot{x}+Y\delta\ddot{y}+Z\delta\ddot{z}-m\,(\ddot{x}\delta\ddot{x}+\ddot{y}\delta\ddot{y}+\ddot{z}\delta\ddot{z})]=0$$

for all values of $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ which are possible and reversible before the addition of the new constraints. But values proportional to $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$, and of the same sign, are evidently consistent with the original constraints, and when the components of acceleration are altered to $\ddot{x}+\ddot{x}'$, $\ddot{y}+\ddot{y}'$, $\ddot{z}+\ddot{z}'$, variations of these quantities proportional to and of the same sign as $-\ddot{x}'$, $-\ddot{y}'$, $-\ddot{z}'$ are evidently consistent with the original constraints. Now, if these latter variations were not possible before the accelerations were modified by the addition of the new forces and constraints, it must be that some constraint was then operative which afterwards ceased to be so. The expression (22) will, therefore, be equal to zero, provided only that all the constraints which were operative before the addition of the new forces and constraints, remain operative afterwards.* With this limitation, therefore, the expression (23) must have the greatest value consistent with the constraints. This principle may be expressed without reference to rectangular coordinates. If we write u' for the relative acceleration due to the additional forces and constraints, we have

$$u'^2=\ddot{x}'^2+\ddot{y}'^2+\ddot{z}'^2,$$

and expression (23) reduces to

(24) $$\Sigma\left(X\ddot{x}'+Y\ddot{y}'+Z\ddot{z}'-\frac{1}{2}\,mu'^2\right).$$

If the sum of the moments of the additional forces which are considered is represented by $\$\,(Qdq)$, (the q representing quantities determined by the configuration of the system,) we have

$$\Sigma\,(X'\dot{x}+Y'\dot{y}+Z'\dot{z})=\$\,(Q\dot{q})\,.$$

We may distinguish the values of $\frac{d^2q}{dt^2}$ immediately before and immediately

*As an illustration of the significance of this limitation, we may consider the condition afforded by the impenetrability of two bodies in contact. Let us suppose that if subject only to the original forces and constraints they would continue in contact, but that, under the influence of the additional forces and constraints, the contact will cease. The impenetrability of the bodies then ceases to be operative as a constraint. Such cases form an exception to the principle which is to be established. But there are no exceptions when all the original constraints are expressed by *equations.*

after the application of the additional forces and constraints by the expressions $\ddot{q}$, and $\ddot{q} + \ddot{q}'$. With this understanding, we have, by differentiation of the preceding equation,

$$\Sigma\left[\dot{X}'\ddot{x} + \dot{Y}'\ddot{y} + \dot{Z}'\ddot{z} + X'(\ddot{x} + \ddot{x}') + Y'(\ddot{y} + \ddot{y}') + Z'(\ddot{z} + \ddot{z}')\right]$$
$$= \$\left[\dot{Q}\ddot{q} + Q(\ddot{q} + \ddot{q}')\right];$$

whence it appears that $\Sigma(X'\ddot{x}' + Y'\ddot{y}' + Z'\ddot{z}')$ differs from $\$(Q\ddot{q}')$ only by quantities which are independent of the relative accelerations due to the additional forces and restraints. It follows that these relative accelerations are such as to make

$$(25) \qquad \$(Q\ddot{q}') - \Sigma\left(\frac{1}{2}mu'^2\right)$$

a maximum.

It will be observed that the condition which determines these relative accelerations is of precisely the same form as that which determines absolute accelerations.

An important case is that in which new constraints are added but no new forces. The relative accelerations are determined in this case by the condition that $\Sigma\left(\frac{1}{2}mu'^2\right)$ is a minimum. In any case of motion, in which finite forces do not act at points, lines or surfaces, we may first calculate the accelerations which would be produced if there were no constraints, and then determine the relative accelerations due to the constraints by the condition that $\Sigma\left(\frac{1}{2}mu'^2\right)$ is a minimum. This is Gauss's principle of *least constraint.**

Again, in any case of motion, we may suppose u to denote the acceleration which would be produced by the constraints alone, and u' the relative acceleration produced by the forces; we then have

$$\Sigma\left[m(\ddot{x}\ddot{x}' + \ddot{y}\ddot{y}' + \ddot{z}\ddot{z}')\right] = 0,$$

whence, if we write u'' for the resultant or actual acceleration,

$$\Sigma\left(\frac{1}{2}mu^2\right) + \Sigma\left(\frac{1}{2}mu'^2\right) = \Sigma\left(\frac{1}{2}mu''^2\right).$$

Moreover, differentiating (25), we obtain

$$\$(Q\delta\ddot{q}') - \Sigma\left[m(\ddot{x}'\delta\ddot{x}' + \ddot{y}'\delta\ddot{y}' + \ddot{z}'\delta\ddot{z}')\right] = 0,$$

*This principle may be derived very directly from the general formula (6), or *vice versa*, for $\Sigma\left(\frac{1}{2}mu'^2\right)$ may be put in the form

$$\Sigma\left[\frac{1}{2}m\left\{\left(\ddot{x} - \frac{X}{m}\right)^2 + \left(\ddot{y} - \frac{Y}{m}\right)^2 + \left(\ddot{z} - \frac{Z}{m}\right)^2\right\}\right],$$

the variation of which, with the sign changed, is identical with the first member of (6).

whence, since $\delta\ddot{q}'$, $\delta\ddot{x}'$, $\delta\ddot{y}'$, $\delta\ddot{z}'$ may have values proportional to $\ddot{q}'$, $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$,

$$\mathfrak{S}\,(Q\ddot{q}') = 2\Sigma\left(\frac{1}{2}\,m\dot{u}'^2\right).$$

These relations are similar to those which exist with respect to *vis viva* and impulsive forces.

Particular Equations of Motion.

From the general formula (12), we may easily obtain particular equations which will express the laws of motion in a very general form.

Let $d\omega_1$, $d\omega_2$, etc., be infinitesimals (not necessarily complete differentials) the values of which are independent, and by means of which we can perfectly define any infinitesimal change in the configuration of the system; and let

$$\dot{\omega}_1 = \frac{d\omega_1}{dt}, \quad \dot{\omega}_2 = \frac{d\omega_2}{dt}, \quad \text{etc.},$$

where $d\omega_1$, $d\omega_2$ are to be determined by the change in the configuration in the interval of time dt; and let

$$\ddot{\omega}_1 = \frac{d\dot{\omega}_1}{dt}, \quad \ddot{\omega}_2 = \frac{d\dot{\omega}_2}{dt}, \quad \text{etc.}$$

Also let

$$U = \Sigma\left(\frac{1}{2}\,m\dot{u}^2\right)$$

It is evident that U can be expressed in terms of $\dot{\omega}_1$, $\dot{\omega}_2$, etc., $\ddot{\omega}_1$, $\ddot{\omega}_2$, etc., and the quantities which express the configuration of the system, and that (since δ is used to denote a variation which does not affect the configuration or the velocities),

$$\delta U = \frac{dU}{d\ddot{\omega}_1}\,\delta\ddot{\omega}_1 + \frac{dU}{d\ddot{\omega}_2}\,\delta\ddot{\omega}_2 + \text{etc.}$$

Moreover, since the quantities p in the general formula are entirely determined by the configuration of the system

$$\dot{p} = \frac{dp}{d\omega_1}\,\dot{\omega}_1 + \frac{dp}{d\omega_2}\,\dot{\omega}_2 + \text{etc.},$$

where $\frac{dp}{d\omega_1}$ denotes the ratio of simultaneous values of dp and $d\omega_1$, when $d\omega_2$ etc., are equal to zero, and $\frac{dp}{d\omega_2}$, etc., are to be interpreted on the same principle. Multiplying by P, and taking the sum with respect to the several forces, we have

$$\mathfrak{S}\,(P\dot{p}) = \Omega_1\dot{\omega}_1, + \Omega_2\dot{\omega}_2 + \text{etc.},$$

where

$$\Omega_1 = \mathfrak{S}\left(P\,\frac{dp}{d\omega_1}\right), \quad \Omega_2 = \mathfrak{S}\left(P\,\frac{dp}{d\omega_2}\right), \quad \text{etc.}$$

If we differentiate with respect to t, and take the variation denoted by δ, we obtain

$$\mathfrak{S}\,(P\delta\ddot{p}) = \Omega_1\delta\ddot{\omega}_1 + \Omega_2\delta\ddot{\omega}_2 + \text{etc.}$$

The general formula (12) is thus reduced to the form

$$(26)\qquad \left(\Omega_1 - \frac{dU}{d\ddot{\omega}_1}\right)\delta\ddot{\omega}_1 + \left(\Omega_2 - \frac{dU}{d\ddot{\omega}_2}\right)\delta\ddot{\omega}_2 + \text{etc.} \geqq 0\,.$$

If the forces have a potential V, we may write

$$(27)\qquad \left(\frac{dV}{d\omega_1} - \frac{dU}{d\ddot{\omega}_1}\right)\delta\ddot{\omega}_1 + \left(\frac{dV}{d\omega_2} - \frac{dU}{d\ddot{\omega}_2}\right)\delta\ddot{\omega}_2 + \text{etc.},$$

where $\frac{dV}{d\omega_1}$ denotes the ratio of dV and $d\omega_1$ when $d\omega_2$, etc., have the value zero, and the analogous expressions are to be interpreted on the same principle.

If the variations $\delta\omega_1$, $\delta\omega_2$, etc., are capable both of positive and of negative values, we must have

$$(28)\qquad \frac{dU}{d\ddot{\omega}_1} = \Omega_1\,, \quad \frac{dU}{d\ddot{\omega}_2} = \Omega_2\,, \quad \text{etc.,}$$

or,

$$(29)\qquad \frac{dU}{d\ddot{\omega}_1} = \frac{dV}{d\omega_1}\,, \quad \frac{dU}{d\ddot{\omega}_2} = \frac{dV}{d\omega_2}\,, \quad \text{etc.}$$

To illustrate the use of these equations in a case in which $d\omega_1$, $d\omega_2$, etc., are not exact differentials, we may apply them to the problem of the rotation of a rigid body of which one point is fixed. If $d\omega_1$, $d\omega_2$, $d\omega_3$ denote infinitesimal rotations about the principal axes which pass through the fixed point, Ω_1, Ω_2, Ω_3 will denote the moments of the impressed forces about these axes, and the value of U will be given by the formula

$$\begin{aligned} 2U = {} & (a+b+c)(\dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2)^2 - (\dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2)(a\dot{\omega}_1^2 + b\dot{\omega}_2^2 + c\dot{\omega}_3^2) \\ & + 2\,(b-c)\,\dot{\omega}_2\dot{\omega}_3\ddot{\omega}_1 + 2\,(c-a)\,\dot{\omega}_3\dot{\omega}_1\ddot{\omega}_2 + 2\,(a-b)\,\dot{\omega}_1\dot{\omega}_2\ddot{\omega}_3 \\ & + (b+c)\,\ddot{\omega}_1^2 + (c+a)\,\ddot{\omega}_2^2 + (a+b)\,\ddot{\omega}_3^2\,, \end{aligned}$$

where a, b, and c are constants, $a+b$, $b+c$, $c+a$ being the *moments of inertia* about the three axes. Hence,

$$\frac{dU}{d\ddot{\omega}_1} = (b-c)\,\dot{\omega}_2\dot{\omega}_3 + (b+c)\,\ddot{\omega}_1, \quad \frac{dU}{d\ddot{\omega}_2} = (c-a)\,\dot{\omega}_3\dot{\omega}_1 + (c+a)\,\ddot{\omega}_2,$$

$$\frac{dU}{d\ddot{\omega}_3} = (a-b)\,\dot{\omega}_1\dot{\omega}_2 + (a+b)\,\ddot{\omega}_3;$$

and the equations of motion are

$$\ddot{\omega}_1 = \frac{(c-b)\,\dot{\omega}_2\dot{\omega}_3 + \Omega_1}{c+b},$$

$$\ddot{\omega}_2 = \frac{(a-c)\,\dot{\omega}_3\dot{\omega}_1 + \Omega_2}{a+c},$$

$$\ddot{\omega}_3 = \frac{(b-a)\,\dot{\omega}_1\dot{\omega}_2 + \Omega_3}{b+a}.$$